Factorization and superpotential of the PT symmetric Hamiltonian

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Abstract
We study the factorization of the PT symmetric Hamiltonian. The general expression for the superpotential corresponding to the PT symmetric potential is obtained and explicit examples are presented.

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1. Introduction

The PT symmetric complex potentials suggested in [1–3] have recently attracted a fair amount of attention. It was shown that for several PT symmetric complex potentials the spectrum of the corresponding Hamiltonian is real so long as the PT symmetry is not spontaneously broken. On its own, this feature gives rise to the main reason for interest in them. Recently it has been proved that supersymmetric (SUSY) quantum mechanics (for a review see [4]) is a useful tool for the investigation of the eigenvalue problem not only for the Hermitian Hamiltonian but also for the non-Hermitian Hamiltonian with a complex potential [5–13].

In this paper we shall answer the following question. What general expression for the superpotential leads to the PT symmetric potentials? This gives us the possibility of obtaining a general expression for the quasi-exactly solvable (QES) PT symmetric potential for which we know, in explicit form, one eigenstate. In this connection it is worth noting that the nature of PT symmetric QES potentials of a special form has been studied in [14–16] (see also references therein).

Note that there is no problem to generate the QES arbitrary complex potential with one known eigenstate. Even for the case of two or three eigenstates it is possible to obtain a general expression for a complex QES potential using the SUSY method proposed in [17, 18] when the PT symmetry is not imposed. But when the PT symmetry is imposed on the potential it is not then a trivial problem to obtain a general expression for the QES potential even with one known eigenstate. Only this problem is considered in this paper.

2. The superpotential of the PT symmetric potentials

Let the Hamiltonian read as

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + V(x) \tag{1}$$

where $V(x) = V_1(x) + iV_2(x)$ is a complex potential. The Hamiltonian is called PT symmetric when

$$PTH = HPT \tag{2}$$

where P is the parity operator acting as the spatial reflection $Pf(x) = f(-x)$, and T is the complex conjugation operator $Tf(x) = f^*(x)$. In explicit form the condition for the PT symmetry for a potential $V(x)$ reads

$$V^*(-x) = V(x). \tag{3}$$

Suppose that the Hamiltonian can be written in the factorized form

$$H = \frac{1}{2}\left(-\frac{d}{dx} + W\right)\left(\frac{d}{dx} + W\right) + \varepsilon = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}(W^2 - W') + \varepsilon \tag{4}$$

where W is the so-called superpotential, $W' = dW/dx$, ε is the energy of factorization. Note that in our case W and ε are complex valued. The wavefunction corresponding to the energy ε reads

$$\psi_\varepsilon = Ce^{-\int W dx}. \tag{5}$$

Using (1) and (4) we obtain the relation between the potential and the superpotential

$$V(x) = \frac{1}{2}(W^2 - W') + \varepsilon. \tag{6}$$

Then the PT symmetry condition (3) reads

$$(W^*(-x))^2 + \frac{d}{dx}W^*(-x) + 2\varepsilon^* = W^2(x) - \frac{d}{dx}W(x) + 2\varepsilon. \tag{7}$$

Thus the potential $V(x)$ is PT symmetric when the superpotential satisfies condition (7). To solve this equation we rewrite it in the following form:

$$U_+'(x) = U_+(x)U_-(x) + 2(\varepsilon - \varepsilon^*) \tag{8}$$

where

$$U_+(x) = W(x) + W^*(-x) \tag{9}$$
$$U_-(x) = W(x) - W^*(-x). \tag{10}$$

As follows from (9) and (10) U_+ is PT symmetric and U_- is anti-PT symmetric:

$$U_+^*(-x) = U_+(x) \tag{11}$$
$$U_-^*(-x) = -U_-(x). \tag{12}$$

Equation (8) can be easily solved with respect to U_- for a given U_+ or vice versa. We use the solution with respect to U_-, i.e.,

$$U_- = \frac{U'_+ - 2(\varepsilon - \varepsilon^*)}{U_+}. \tag{13}$$

Then from (9) and (10) we obtain

$$W = \frac{1}{2}\left\{U_+ + \frac{U'_+ - 2(\varepsilon - \varepsilon^*)}{U_+}\right\}. \tag{14}$$

This expression for the superpotential is the main result of this paper. It is interesting to note that equation (8) and the expression for superpotential (14) are formally similar to the equation and superpotential obtained in our papers [17, 18] where we studied the real QES potential with two known eigenstates.

Substituting (14) into (6) we obtain the PT symmetric potential which can be written in the following form:

$$V(x) = \tfrac{1}{8}(U_+^2 + U_-^2) - \tfrac{1}{4}U_-' + \tfrac{1}{2}(\varepsilon^* + \varepsilon). \tag{15}$$

Note that for this potential we know in the explicit form at least one level ε and the corresponding wavefunction (5). This function corresponds to the discrete spectrum when it vanishes at infinity or to the continuum spectrum when it is restricted. In these cases potential (15) can be called the PT symmetric QES potential with one known eigenstate. It is also possible that (5) will not satisfy the necessary conditions. Then this function does not belong to the eigenfunctions of the Hamiltonian.

Now let us consider the case of the PT symmetric wavefunction, namely, $PT\psi_0 = \psi_0$. In this case we have

$$W^*(-x) = -W(x) \tag{16}$$

and thus $U_+ = 0$. Therefore, in order to use equation (14) for this case we put $U_+ = \alpha f(x)$, where α tends to zero. Then from (14) we obtain

$$W(x) = \frac{1}{2}\left\{\frac{f'}{f} - B\frac{\mathrm{i}}{f}\right\} \tag{17}$$

where

$$B = \lim_{\alpha \to 0} 2\frac{\varepsilon - \varepsilon^*}{\alpha}.$$

We see that the imaginary part of the energy must tend to zero. Thus, we show that (14) in the special case (16) reproduces the well known result. Namely when the wavefunction is PT symmetric the eigenvalue is real.

3. Examples

To illustrate the described method we give two explicit examples of the PT symmetric potentials. All expressions depend on the function $U_+(x)$, which can be called a generating function. We may choose various functions $U_+(x)$ and obtain as a result various PT symmetric potentials. In the considered examples we specially choose such a function $U_+(x)$ which leads to the proper eigenfunction. Therefore, the obtained PT symmetric potentials are the QES ones with one known eigenstate.

Example 1. Let us consider

$$U_+ = \frac{\mathrm{i}\alpha}{(x + \mathrm{i}a)^n} \tag{18}$$

where n is an odd number. Then

$$U_- = -\frac{n}{x + \mathrm{i}a} - \frac{4\mathrm{Im}\,\varepsilon}{\alpha}(x + \mathrm{i}a)^n \tag{19}$$

$$W = \frac{\mathrm{i}\alpha}{2(x + \mathrm{i}a)^n} - \frac{n}{2(x + \mathrm{i}a)} - \frac{2\mathrm{Im}\,\varepsilon}{\alpha}(x + \mathrm{i}a)^n \tag{20}$$

$$V = \mathrm{Re}\,\varepsilon - \frac{\alpha^2}{8}\frac{1}{(x + \mathrm{i}a)^{2n}} + 2\frac{(\mathrm{Im}\,\varepsilon)^2}{\alpha^2}(x + \mathrm{i}a)^{2n} + \frac{1}{8}\frac{n^2 - 2n}{(x + \mathrm{i}a)^2} + 2\frac{n\mathrm{Im}\,\varepsilon}{\alpha}(x + \mathrm{i}a)^{n-1}. \tag{21}$$

If $n > 1$ then we can write the wavefunction in the following form:

$$\psi_\varepsilon = C(x + ia)^{n/2} \exp\left(\frac{i\alpha}{2(n-1)} \frac{1}{(x+ia)^{n-1}} + 2\frac{\text{Im}\,\varepsilon}{\alpha} \frac{(x+ia)^{n+1}}{n+1}\right) \qquad (22)$$

or if $n = 1$, it reads

$$\psi_\varepsilon = C(x + ia)^{(1-i\alpha)/2} \exp\left(\frac{\text{Im}\,\varepsilon}{\alpha}(x+ia)^2\right). \qquad (23)$$

To obtain a bound state we must set $\frac{\text{Im}\,\varepsilon}{\alpha} < 0$.

Note that in the case of $n = 1$ we have the PT symmetric harmonic oscillator with a regularized centrifugal-like core which is exactly solvable [19, 20]. Thus, the considered example generalizes the PT symmetric harmonic oscillator to the QES PT symmetric potential with one known eigenstate.

It is interesting to stress that in the limit $\text{Im}\,\varepsilon \to 0$, $\alpha \to 0$ and $\lim 2\frac{\text{Im}\,\varepsilon}{\alpha} = B = \text{const}$ we obtain the PT symmetric wavefunction with a real eigenvalue. This just confirms the result obtained in the end of the previous section.

Example 2. This example represents the periodic PT symmetric potential. Let us take

$$U_+ = \alpha e^{ikx}. \qquad (24)$$

Then we obtain

$$U_- = ik - 4i\frac{\text{Im}\,\varepsilon}{\alpha} e^{-ikx} \qquad (25)$$

and the superpotential, potential and wavefunction read, respectively,

$$W = \frac{\alpha}{2}e^{ikx} + \frac{ik}{2} - 2i\frac{\text{Im}\,\varepsilon}{\alpha} e^{-ikx} \qquad (26)$$

$$V = \text{Re}\,\varepsilon - \frac{k^2}{8} + \frac{\alpha^2}{8}e^{2ikx} - 2\frac{(\text{Im}\,\varepsilon)^2}{\alpha^2}e^{-2ikx} + 2k\frac{\text{Im}\,\varepsilon}{\alpha}e^{-ikx} \qquad (27)$$

$$\psi_\varepsilon = \exp\left(-\frac{ikx}{2} + \frac{i\alpha}{2k}e^{ikx} - 2\frac{\text{Im}\,\varepsilon}{\alpha k}e^{-ikx}\right). \qquad (28)$$

In the case of $\text{Im}\,\varepsilon = 0$ this QES potential becomes exactly solvable and corresponds to the potential studied in [5].

4. Conclusions

We have obtained the general expression for superpotential (14) which corresponds to the PT symmetric potential (15). The PT symmetric function $U_+(x)$ plays the role of the generating function. Choosing different functions $U_+(x)$ we obtain different superpotentials (14) and the corresponding PT symmetric potentials (6) or (15). These PT symmetric potentials can be considered as QES potentials with one known eigenfunction (5) and the corresponding energy ε. Of course, the solution (5) must satisfy necessary conditions in order to be the eigenfunction of the Hamiltonian.

In the considered examples we specifically choose the $U_+(x)$ which lead to proper eigenfunctions. The potentials considered in examples 1 and 2 are interesting because at some values of the parameters they become exactly solvable potentials which were studied earlier. Thus, the considered examples generalize the exactly solvable PT symmetric potentials to the QES potentials with one known eigenstate.

References

[1] Bender C M and Boettcher S 1998 *Phys. Rev. Lett.* **80** 5243
[2] Bender C M, Boettcher S and Meisinger P N 1999 *J. Math. Phys.* **40** 2201
[3] Bender C M 1999 *Phys. Rep.* **315** 27
[4] Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **251** 267
[5] Cannata F, Junker G and Trost J 1998 *Phys. Lett.* A **246** 219
[6] Andrianov A A, Cannata F, Denonder J-P and Ioffe M V 1999 *Int. J. Mod. Phys.* A **14** 2675
[7] Znojil M, Cannata F, Bagchi B and Roychoudhury R 2000 *Phys. Lett.* B **483** 284
[8] Bagchi B, Cannata F and Quesne C 2000 *Phys. Lett.* A **269** 79
[9] Bagchi B and Roychoudhury R 2000 *J. Phys. A: Math. Gen.* **33** L1
[10] Cannata F, Ioffe M, Roychoudhury R and Roy P 2001 *Phys. Lett.* A **281** 305
[11] Znojil M 2001 *Czech. J. Phys.* **51** 420
     (Znojil M 2001 *Preprint* hep-ph/0101038)
[12] Bagchi B, Mallik S and Quesne C 2001 *Int. J. Mod. Phys.* A **16** 2859
     (Bagchi B, Mallik S and Quesne C 2001 *Preprint* quant-ph/0102093)
[13] Dorey P, Dunning C and Tateo R 2001 *J. Phys. A: Math. Gen.* **34** L391
[14] Bender C M and Boettcher S 1998 *J. Phys. A: Math. Gen.* **31** L273
[15] Khare A and Mandal B P 2000 *Phys. Lett.* A **272** 53
[16] Bagchi B, Mallik S, Quesne C and Roychoudhury R 2001 *Preprint* quant-ph/0107095
[17] Tkachuk V M 1998 *Phys. Lett.* A **245** 177
[18] Tkachuk V M 1999 *J. Phys. A: Math. Gen.* **32** 1291
[19] Znojil M 1999 *Phys. Lett.* A **259** 220
[20] Lévai G and Znojil M 2000 *J. Phys. A: Math. Gen.* **33** 7165